J. Math. Anal. Appl. 387 (2012) 284–290

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bresse system with indefinite damping

Juan A. Soriano [a,*], Jaime E. Muñoz Rivera [b,c], Luci Harue Fatori [d]

[a] *Department of Mathematics, Universidade Estadual de Maringá, Campus Universitário, Maringá, CEP 87020-900, PR, Brazil*
[b] *National Laboratory of Scientific Computations, LNCC/MCT, Rua Getúlio Vargas 333, Quitandinha, Petrópolis, CEP 25651-070, RJ, Brazil*
[c] *Institute of Mathematics, UFRJ, Brazil*
[d] *Department of Mathematics, Universidade Estadual de Londrina, Campus Universitário, Londrina, CEP 86.051-990, PR, Brazil*

ARTICLE INFO

Article history:
Received 1 May 2011
Available online 3 September 2011
Submitted by D.L. Russell

Keywords:
Bresse system
Indefinite damping
Exponential stability

ABSTRACT

We consider the Bresse system in a bounded domain $(0, L) \subset \mathbb{R}^1$. The system has an indefinite damping mechanism, i.e. with a damping function $a = a(x)$ possibly changing sign, presented only in the equation for the rotation angle. We shall prove that the system is still exponentially stable under the same conditions as in the positive constant damping case, and provided $\bar{a} = \frac{1}{L}\int_0^L a(x)\,dx > 0$ and $\|a - \bar{a}\|_{L^2} < \epsilon$, for ϵ small enough. In the arguments, we shall also give a new proof of exponential stability for the constant case $a = \bar{a}$.

1. Introduction

In this paper we consider the Bresse system with indefinite damping mechanism presented only in the equation of the system. More specifically we will consider the following system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0 \quad \text{in }]0,\infty[\times]0,L[, \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + a(x)\psi_t = 0 \quad \text{in }]0,\infty[\times]0,L[, \tag{1.2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0 \quad \text{in }]0,\infty[\times]0,L[, \tag{1.3}$$

where the constants ρ_1, k, ρ_2, b, l, k_0 are positive. We consider the following initial conditions

$$\varphi(0,\cdot) = \varphi_0, \qquad \varphi_t(0,\cdot) = \varphi_1, \qquad \psi(0,\cdot) = \psi_0, \qquad \psi_t(0,\cdot) = \psi_1,$$
$$w(0,\cdot) = w_0, \qquad w_t(0,\cdot) = w_1,$$

and Dirichlet–Neumann–Neumann boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = w_x(t,0) = w_x(t,L) = 0 \quad \text{in }]0,\infty[. \tag{1.4}$$

The damping mechanism is present in Eq. (1.2), given by $a(x)\psi_t$, where $a \in L^\infty(0,L)$ may change sign, but satisfies

$$\bar{a} = \frac{1}{L}\int_0^L a(x)\,dx > 0. \tag{1.5}$$

* Corresponding author.
E-mail address: jaspalomino@uem.br (J.A. Soriano).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.08.072

There exist a few results about the asymptotic behavior the Bresse system where the authors consider the different kinds of the dissipative mechanism. For example, in [7] Z. Liu and B. Rao consider the Bresse system with two different dissipative mechanisms, given by two heat equations, non-dissipative coupled to the system. In [9] the authors consider the Bresse system with thermal dissipation effective in one equation of the system. Regarding the non-dissipative case with indefinite a we can mention the works of Chen et al. [3], Freitas [4], Freitas and Zuazua [5], Benaddi and Rao [1], Liu and Rao [6], Menz [8], Muñoz and Racke [10] among others.

Our main result is to prove that the system (1.1)–(1.4) is exponentially stable when a verifies (1.5) and $\|a-\bar{a}\|_{L^2}$ is small and the wave speeds are equal, that is,

$$\frac{\rho_1}{\rho_2}=\frac{k}{b} \quad \text{and} \quad k=k_0. \tag{1.6}$$

The method we use to show the exponential stability is based on *Gearhart–Herbst–Prüß–Huang* theorem to dissipative systems and fixed point arguments.

Theorem 1.1. *Let $S(t)=e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A})\supseteq\{i\beta\colon\ \beta\in\mathbb{R}\}\equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\left\|(i\beta I-\mathcal{A})^{-1}\right\|<\infty$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

To no dissipative systems we used the following result

Theorem 1.2. *Let $S(t)=e^{\mathcal{A}t}$ be a C_0-semigroup on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A})\supseteq\{\lambda\colon\ Re\,\lambda\geqslant 0\}$$

and

$$\left\|(\lambda I-\mathcal{A})^{-1}\right\|<M \quad \textit{for all } Re\,\lambda\geqslant 0$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.

The paper is organized as follows. In Section 2 we will formulate the semigroup setting. In Section 3 we will show the exponential stability.

2. The semigroup setting

We rewrite the initial–boundary value problem (1.1)–(1.4) as a first-order system for $U:=(\varphi,\varphi_t,\psi,\psi_t,w,w_t)'$, where the prime is used to denote the transpose. Then U satisfies

$$U_t=\mathcal{A}U,\qquad U(0)=U_0, \tag{2.1}$$

where $U_0:=(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1)'$ and $\mathcal{A}$ is the (formal) differential operator

$$\mathcal{A}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ k/\rho_1\partial_x^2-k_0l^2I/\rho_1 & 0 & k/\rho_1\partial_x & 0 & (k+k_0)/\rho_1 l\partial_x & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ -k/\rho_2\partial_x & 0 & b/\rho_2\partial_x^2-k/\rho_2 I & -a(x)/\rho_2 I & -kl/\rho_2 I & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -(k_0+k)l/\rho_1\partial_x & 0 & -lkI/\rho_1 & 0 & k_0/\rho_1\partial_x^2-l^2kI & 0\end{pmatrix}. \tag{2.2}$$

We introduce the Hilbert space

$$\mathcal{H}:=H_0^1(0,L)\times L^2(0,L)\times H_*^1(0,L)\times L_*^2(0,L)\times H_*^1(0,L)\times L_*^2(0,L)$$

with

$$L^2_*(0,L)=\left\{f\in L^2(0,l);\ \int_0^L f\,dx=0\right\},$$

$$H^1_*(0,L)=H^1(0,L)\cap L^2_*(0,L),$$

and norm given by

$$\begin{aligned}\|U\|^2_{\mathcal{H}}&=\left\|(\varphi,\Phi,\psi,\Psi,w,\mathrm{W})'\right\|^2_{\mathcal{H}}\\&\equiv\int_0^L \rho_1|\Phi|^2+\rho_2|\Psi|^2+\rho_1|\mathrm{W}|^2+b|\psi_x|^2+k|\varphi_x+\psi+lw|^2+k_0|w_x-l\varphi|^2\,dx.\end{aligned}$$

Then $\mathcal{A}$, formally given in (2.2), with domain

$$D(\mathcal{A}):=\left\{U\in\mathcal{H}\;\middle|\;\varphi\in H^2(0,L)\cap H^1_0(0,L),\ \Phi\in H^1_0(0,L),\ \Psi,\mathrm{W}\in H^1_*(0,L),\ \psi_x,w_x\in H^1_0(0,L)\right\},$$

generates a semigroup $\{e^{tA}\}_{t\geqslant 0}$. We observe that for a solution (φ,ψ,w) to (1.1)–(1.4) and the corresponding U, the norm $\|U(t)\|^2_{\mathcal{H}}$ equals twice the energy $E(t)$ of (φ,ψ,ω) defined by

$$E(t):=\frac{1}{2}\int_0^L\big(\rho_1|\Phi|^2+\rho_2|\Psi|^2+\rho_1|\mathrm{W}|^2+b|\psi_x|^2+k|\varphi_x+\psi+lw|^2+k_0|w_x-l\varphi|^2\big)(t,x)\,dx. \tag{2.3}$$

Replacing the function $a=a(x)$ in (1.2) by the constant $\bar{a}$, we write $\mathcal{A}_c$ for the arising constant coefficient operator instead of $\mathcal{A}$, i.e.,

$$\mathcal{A}_c=\begin{pmatrix}0&1&0&0&0&0\\ k/\rho_1\partial_x^2-k_0l^2I/\rho_1&0&k/\rho_1\partial_x&0&(k+k_0)/\rho_1 l\partial_x&0\\ 0&0&0&1&0&0\\ -k/\rho_2\partial_x&0&b/\rho_2\partial_x^2-k/\rho_2 I&-\bar{a}/\rho_2 I&-kl/\rho_2 I&0\\ 0&0&0&0&0&1\\ -(k_0+k)l/\rho_1\partial_x&0&-lkI/\rho_1&0&k_0/\rho_1\partial_x^2-l^2kI&0\end{pmatrix}. \tag{2.4}$$

3. Main result

The stability of the system (1.1)–(1.4) is based on the result [2], which proves the exponential stability for the constant coefficients case.

Theorem 3.1. *The semigroup associated to the system* (1.1)–(1.4) *is exponentially stable if and only if*

$$\frac{\rho_1}{\rho_2}=\frac{k}{b}\quad\textit{and}\quad k=k_0.$$

Remark 3.2. In view of Theorem 1.2 the resolvent operator for constant coefficients verifies

$$\left\|(\lambda I-\mathcal{A}_c)^{-1}\right\|\leqslant c\quad\forall Re\,\lambda\geqslant 0.$$

Let $F\in\mathcal{H}$ and $U\in D(\mathcal{A})$. We introduce the resolvent equation

$$(\lambda-\mathcal{A})U=F$$

which is equivalent to solving

$$\begin{aligned}(\lambda-\mathcal{A}_c)U&=F+(\mathcal{A}-\mathcal{A}_c)U\\&=F-(a-\bar{a})BU,\end{aligned} \tag{3.1}$$

with

$$B=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1/\rho_2 I&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}. \tag{3.2}$$

If $F=(F_1,F_2,F_3,F_4,F_5,F_6)'$ and $U=(\varphi,\Phi,\psi,\Psi,w,\mathrm{W})'$ then we rewrite (3.1) as

$$\lambda\varphi-\Phi=F_1\in H_0^1(0,L), \tag{3.3}$$

$$\rho_1\lambda\Phi-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)=\rho_1F_2\in L^2(0,L), \tag{3.4}$$

$$\lambda\psi-\Psi=F_3\in H_*^1(0,L), \tag{3.5}$$

$$\rho_2\lambda\Psi-b\psi_{xx}+k(\varphi_x+\psi+lw)+\bar a\Psi=\rho_2F_4-(a-\bar a)\Psi\in L_*^2(0,L), \tag{3.6}$$

$$\lambda w-\mathrm{W}=F_5\in H_*^1(0,L), \tag{3.7}$$

$$\rho_1\mathrm{W}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)=\rho_1F_6\in L_*^2(0,L). \tag{3.8}$$

Replacing Φ, Ψ, W given by (3.3), (3.5), (3.7) into (3.4), (3.6) and (3.8), respectively, and using the Dirichlet–Neumann–Neumann boundary condition, we get

$$\rho_1\lambda^2\varphi-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)=f_1, \tag{3.9}$$

$$\rho_2\lambda^2\psi-b\psi_{xx}+k(\varphi_x+\psi+lw)+\bar a\lambda\psi=(\bar a-a)\lambda\psi+f_2, \tag{3.10}$$

$$\rho_1\lambda^2w-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)=f_3, \tag{3.11}$$

$$\varphi(0)=\varphi(L)=\psi_x(0)=\psi_x(L)=w_x(0)=w_x(L)=0, \tag{3.12}$$

where

$$f_1=\rho_1(\lambda F_1+F_2),\qquad f_2=(\rho_2\lambda+a)F_3+\rho_2F_4,\qquad f_3=\rho_1(\lambda F_5+F_6).$$

On the other hand, (3.10) can be rewritten as

$$\psi_{xx}-\underbrace{\left(\frac{\rho_2\lambda^2+\bar a\lambda+k}{b}\right)}_{:=\alpha^2}\psi=\frac{k}{b}(\varphi_x+lw)+\frac{a-\bar a}{b}\lambda\psi-\frac{1}{b}f_2. \tag{3.13}$$

We introduce the operator $\mathcal{N}_\alpha$ such that $\mathcal{N}_\alpha(g)$ denotes the solution v of the Neumann problem

$$v_{xx}-\alpha^2v=g,\qquad v_x(0)=v_x(L)=0, \tag{3.14}$$

that is,

$$v=\mathcal{N}_\alpha(g).$$

Therefore the solution of Eq. (3.13) can be written as

$$\psi=\mathcal{N}_\alpha\left(\frac{k}{b}(\varphi_x+lw)+\frac{a-\bar a}{b}\lambda\psi-\frac{1}{b}f_2\right), \tag{3.15}$$

where

$$\mathcal{N}_\alpha(g)=-\frac{1}{\alpha}\frac{\cosh(\alpha x)}{\sinh(\alpha L)}\int_0^L\cosh\bigl(\alpha(L-s)\bigr)g(s)\,ds+\frac{1}{\alpha}\int_0^L\sinh\bigl(\alpha(x-s)\bigr)g(s)\,ds.$$

Replacing (3.15) in (3.10), we obtain

$$\rho_2\lambda^2\psi-b\psi_{xx}+k(\varphi_x+\psi+lw)+\bar a\lambda\psi=(\bar a-a)\lambda G(\varphi,\psi,w)+f_2 \tag{3.16}$$

where G is given by

$$G(u,v,z)=\frac{k}{b}\mathcal{N}_\alpha(u_x+lz)+\frac{\lambda}{b}\mathcal{N}_\alpha\big((a-\bar{a})v\big)-\frac{1}{b}\mathcal{N}_\alpha(f_2).$$

Let us introduce the operator P

$$\begin{aligned}&P: H_0^1(0,L)\times H_*^1(0,L)\times H_*^1(0,L)\longrightarrow H_0^1(0,L)\times H_*^1(0,L)\times H_*^1(0,L),\\&(u,v,z)\longmapsto(\varphi,\psi,w),\end{aligned}$$

where (φ,ψ,w) is a solution of

$$\rho_1\lambda^2\varphi-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)=f_1, \tag{3.17}$$

$$\rho_2\lambda^2\psi-b\psi_{xx}+k(\varphi_x+\psi+lw)+\bar{a}\lambda\psi=(\bar{a}-a)\lambda G(u,v,z)+f_2, \tag{3.18}$$

$$\rho_1\lambda^2 w-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)=f_3, \tag{3.19}$$

$$\varphi(0)=\varphi(L)=\psi_x(0)=\psi_x(L)=w_x(0)=w_x(L), \tag{3.20}$$

which is well defined since $\lambda\in\varrho(\mathcal{A}_c)$. Let us introduce the norm $\|\cdot\|_\lambda$ given by

$$\big\|(u,v,z)\big\|_\lambda^2:=\int_0^L\rho_1|\lambda u|^2+\rho_2|\lambda v|^2+\rho_1|\lambda z|^2+b|v_x|^2+k|u_x+v+lz|^2+k_0|z_x-lu|^2\,dx.$$

Our next step is to prove that the operator P has a fixed point (φ,ψ,w) which solves the system (3.9)–(3.12). The following lemmas are in order to.

Lemma 3.3. *Assume $\|a-\bar{a}\|_{L^2}<\epsilon$, for $\epsilon>0$ small enough. Under the above notations the fixed point of the operator P defined above is a solution of the system* (3.9)–(3.12).

Proof. Let (φ,ψ,w) be this fixed point, and let

$$\hat{\psi}:=G(\varphi,\psi,w)=\frac{k}{b}\mathcal{N}_\alpha(\varphi_x+lw)+\frac{\lambda}{b}\mathcal{N}_\alpha\big((a-\bar{a})\psi\big)-\frac{1}{b}\mathcal{N}_\alpha(f_2),$$

hence

$$\begin{aligned}&\hat{\psi}_{xx}-\alpha^2\hat{\psi}=\frac{k}{b}(\varphi_x+lw)+\frac{\lambda}{b}(a-\bar{a})\psi-\frac{1}{b}f_2,\\&\hat{\psi}_x(0)=\hat{\psi}_x(L)=0,\end{aligned}$$

implying

$$\rho_2\lambda^2\hat{\psi}-b\hat{\psi}_{xx}+k(\varphi_x+\hat{\psi}+lw)+\bar{a}\lambda\hat{\psi}=\lambda(\bar{a}-a)\psi+f_2. \tag{3.21}$$

Since (φ,ψ,w) is a fixed point of P, we also have

$$\rho_2\lambda^2\psi-b\psi_{xx}+k(\varphi_x+\psi+lw)+\bar{a}\lambda\psi=\lambda(\bar{a}-a)\hat{\psi}+f_2. \tag{3.22}$$

We conclude for the difference $\Psi:=\hat{\psi}-\psi$

$$\Psi_{xx}-\alpha^2\Psi=\frac{\lambda(a-\bar{a})}{b}\Psi,$$

or

$$\Psi=\mathcal{N}_\alpha\left(\frac{\lambda(a-\bar{a})}{b}\Psi\right).$$

Using the estimates of $\mathcal{N}_\alpha$ (see [11]) we conclude that there are positive constants c_1 and c_2, such that

$$|\Psi|\leqslant c_1\big\|(a-\bar{a})\Psi\big\|_{L^1}\leqslant c_1\|a-\bar{a}\|_{L^2}\|\Psi\|_{L^2},$$

where the symbol $|\cdot|$ means the absolute value. Hence

$$\|\Psi\|_{L^2}\leqslant c_2\|a-\bar{a}\|_{L^2}\|\Psi\|_{L^2},$$

implying $\Psi=0$ when $\|a-\bar{a}\|_{L^2}<\frac{1}{c_2}$, therefore $\hat{\psi}=\psi$. $\square$

Taking $f_1 = f_2 = f_3 = 0$ in (3.17)–(3.20) we obtain

$$\rho_1\lambda^2\varphi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0, \tag{3.23}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + \bar{a}\lambda\psi = (\bar{a} - a)\lambda G(u, v, z), \tag{3.24}$$

$$\rho_1\lambda^2 w - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0, \tag{3.25}$$

$$\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = w_x(0) = w_x(L) = 0. \tag{3.26}$$

To prove that P is a contraction we will use the following lemma.

Lemma 3.4. *Under the above notations there exists $C > 0$ such that a solution (φ, ψ, w) of the system* (3.23)–(3.26) *satisfies the following inequality*

$$\left\|(\varphi, \psi, w)\right\|_\lambda \leqslant C\|a - \bar{a}\|_{L^2}\left\|(u, v, z)\right\|_\lambda,$$

for $Re\,\lambda \geqslant 0$.

Proof. Considering the system (3.23)–(3.26) and Remark 3.2 we get

$$\left\|(\varphi, \psi, w)\right\|_\lambda = \|U\|_{\mathcal{H}} \leqslant C\left(\int_0^L \left|(\bar{a} - a)\lambda G(u, v, z)\right|^2 dx\right)^{\frac{1}{2}}. \tag{3.27}$$

Using the estimates of $\mathcal{N}_\alpha$ (see [11]) we conclude that there exists a constant $C > 0$ such that

$$\left|\mathcal{N}_\alpha(u_x + lz)(s)\right| \leqslant \frac{C}{|\lambda|}\|u_x + lz\|_{L^2}, \qquad |\lambda|^2\left|\mathcal{N}_\alpha\big((a - \bar{a})v\big)(s)\right| \leqslant C\|a - \bar{a}\|_{L^2}\|\lambda v\|_{L^2}.$$

Thus

$$\left|\lambda G(u, v, z)(s)\right| \leqslant C\left\|(u, v, z)\right\|_\lambda. \tag{3.28}$$

Combining (3.27) with (3.28) our conclusion follows. □

Lemma 3.5. *Under the above notations and if $\|a - \bar{a}\|_{L^2} < \epsilon$, for $\epsilon > 0$ small enough, then the operator P defined above is a contraction.*

Proof. We will show that there exists a constant $0 < d < 1$, such that

$$\left\|P((u_1 - u_2, v_1 - v_2, z_1 - z_2))\right\|_\lambda \leqslant d\left\|(u_1 - u_2, v_1 - v_2, z_1 - z_2)\right\|_\lambda.$$

Or equivalently that

$$\left\|(\varphi, \psi, w)\right\|_\lambda \leqslant d\left\|(u, v, z)\right\|_\lambda$$

where

$$(\varphi, \psi, w) = (\varphi_1 - \varphi_2, \psi_1 - \psi_2, w_1 - w_2), \qquad (u, v, z) = (u_1 - u_2, v_1 - v_2, z_1 - z_2)$$

satisfies (3.23)–(3.26). Using Lemma 3.4 we get

$$\left\|(\varphi, \psi, w)\right\|_\lambda \leqslant C\|a - \bar{a}\|_{L^2}\left\|(u, v, z)\right\|_\lambda,$$

hence,

$$\left\|(\varphi, \psi, w)\right\|_\lambda \leqslant d\left\|(u, v, z)\right\|_\lambda \tag{3.29}$$

for some $0 < d < 1$ provided $\|a - \bar{a}\|_{L^2}$ is small enough. □

Theorem 3.6 *(Main result). Assume* (1.5) *and* (1.6)*. If $\|a - \bar{a}\|_{L^2} < \epsilon$, for $\epsilon > 0$ small enough, then the semigroup associated to the system* (1.1)–(1.4) *is exponentially stable.*

Proof. From Lemma 3.3 and Lemma 3.5 there exists a unique fixed point (φ, ψ, w) of P solution to (3.9)–(3.12), as proved in Lemma 3.3.

Therefore $U = (\varphi, \lambda\varphi, \psi, \lambda\psi, w, \lambda w)'$ is the unique solution of $(\lambda I - \mathcal{A})U = F$

$$U = (\varphi, \lambda\varphi, \psi, \lambda\psi, w, \lambda w)' + (0, -F_1, 0, -F_3, 0, -F_5)'.$$

Let $\tilde{U}$ be the solution to $(\lambda I - \mathcal{A}_c)\tilde{U} = F$, i.e.

$$\tilde{U} = (\tilde{\varphi}, \lambda\tilde{\varphi}, \tilde{\psi}, \lambda\tilde{\psi}, \tilde{w}, \lambda\tilde{w})' + (0, -F_1, 0, -F_3, 0, -F_5)'.$$

Note that

$$(\tilde{\varphi}, \tilde{\psi}, \tilde{w}) = P(0,0,0).$$

Then we obtain

$$\begin{aligned}
\|U\|_{\mathcal{H}} - \|\tilde{U}\|_{\mathcal{H}} &\leqslant \|U - \tilde{U}\|_{\mathcal{H}} = \big\|(\varphi, \psi, w) - (\tilde{\varphi}, \tilde{\psi}, \tilde{w})\big\|_{\lambda} \\
&= \big\|P((\varphi, \psi, w)) - P((0,0,0))\big\|_{\lambda} \\
&\leqslant d\big\|(\varphi, \psi, w)\big\|_{\lambda} \\
&\leqslant d\|U\|_{\mathcal{H}} + d\|F\|_{\mathcal{H}},
\end{aligned}$$

hence

$$\|U\|_{\mathcal{H}} \leqslant \frac{1}{1-d}\|\tilde{U}\|_{\mathcal{H}} + \frac{d}{1-d}\|F\|_{\mathcal{H}}. \tag{3.30}$$

Since $\lambda \in \varrho(\mathcal{A}_c)$ we obtain

$$\|\tilde{U}\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}. \tag{3.31}$$

From (3.30) and (3.31) we get

$$\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}.$$

Hence

$$\big\|(\lambda I - \mathcal{A})^{-1}F\big\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}, \quad \forall F \in \mathcal{H},$$

which implies

$$\big\|(\lambda I - \mathcal{A})^{-1}\big\|_{\mathcal{H}} \leqslant C, \quad \forall Re\,\lambda \geqslant 0.$$

Thus, by Theorem 1.2 we obtain the exponential stability. □

References

[1] A. Benaddi, B. Rao, Energy decay rate of wave equations with indefinite damping, J. Differential Equations 161 (2000) 337–357.
[2] F.A. Boussouira, J.E. Muñoz Rivera, D. Almeida Jr., Stability to weak dissipative Bresse system, J. Math. Anal. Appl. 374 (2) (2011) 481–498.
[3] G. Chen, S.A. Fulling, F.J. Narcowich, S. Sun, Exponential decay of energy of evolution equations with locally distributed damping, SIAM J. Appl. Math. 51 (1991) 266–301.
[4] P. Freitas, On some eigenvalue problems related to the wave equation with indefinite damping, J. Differential Equations 127 (1996) 320–335.
[5] P. Freitas, E. Zuazua, Stability results for the wave equation with indefinite damping, J. Differential Equations 132 (1996) 338–353.
[6] K. Liu, B. Rao, X. Zhang, Stabilization of the wave equations with potential and indefinite damping, J. Math. Anal. Appl. 269 (2002) 747–769.
[7] Z. Liu, B. Rao, Energy decay rate of the thermoelastic Bresse system, Z. Angew. Math. Phys. 60 (2009) 54–69.
[8] G. Menz, Exponential stability of wave equations with potential and indefinite damping, Konstanzer Schriften Math. Inf. 224 (2007).
[9] L.H. Fatori, J.E. Muñoz Rivera, Rates of decay to weak thermoelastic Bresse system, IMA J. Appl. Math. 75 (1) (2010) 881–904, doi:10.1093/imamat/hxq038.
[10] J.E. Muñoz Rivera, R. Racke, Timoshenko systems with indefinite damping, J. Math. Anal. Appl. 341 (2008) 1068–1083.
[11] J.E. Muñoz Rivera, R. Racke, Exponential stability for wave equations with non-dissipative damping, Nonlinear Anal. 68 (1) (2008) 2531–2551, doi:10.1016/j.na.2007.02.022.